

# Ballistic properties of multilayered concrete shields

G. Ben-Dor, A. Dubinsky, T. Elperin\*

Pearlstone Center for Aeronautical Engineering, Department of Mechanical Engineering, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel

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## ABSTRACT

We investigate theoretically the effect of layering of concrete shields using a wide class of semi-empirical models. We have found that (i) the ballistic limit velocity (BLV) of the multilayered shield does not depend on the order of the plates in the shield, (ii) monolithic shield is superior to any layered shield with the same thickness, and (iii) the largest decrease of the BLV occurs when a shield is divided into a number of plates having the same thickness.

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## 1. Introduction

There is no consensus on the effect of layering on the ballistic properties of shields although this topic has attracted considerable interest among the researchers in the field for a long time. Most of the studies consider the metallic shields; more detailed information can be found in the surveys (Ben-Dor et al., 2005; Corbett et al., 1996), monograph (Ben-Dor et al., 2006a) and some recent publications (Ben-Dor et al., 2006b,c; Dey et al., 2007; Liaghat et al., 2005; Teng et al., 2007, 2008; Zhou and Stronge, 2008).

There are only a few publications on this subject regarding the concrete shields. It is noted in TM 5-855-1 (1986) that a thick slab manufactured of several layers generally has smaller perforation resistance than a single slab with the same thickness. Kojima (1991) performed experiments with a spherical-nosed projectile having velocity 200 m/s and striking the reinforced concrete plates. He tested a monolithic shield with the thickness 18 cm and double-layer shields, 9 cm + 9 cm and 6 cm + 12 cm. The conclusion was that the impact resistance of a double-layer shield is inferior to that of a monolithic shield. Shirai et al. (1997) investigated experimentally and numerically the impact resistance of the reinforced concrete plates against the flat-nosed projectile at impact velocity of about 170 m/s. They compared the protective effectiveness of the monolithic shield with the thickness of 9 cm and double-layer shields having different structure, 4.5 cm + 4.5 cm and 3 cm + 6 cm. It was found that double-layer shields had a higher impact resistance than monolithic shield. Amde et al. (1997) conducted computer simulations using empirical models for concrete and steel plates. They arrived at the conclusions that (i) composite barriers manufactured from different materials are more efficient than layered barriers manufactured from the same material and (ii) multiple barriers are

less efficient than single layer panels having the same thickness.

Approximate models are widely used for estimating protective properties of concrete shields (for more details, see Adeli and Amin, 1985; Bangash, 2009; Guirgis and Guirgis, 2009; Kennedy, 1976; Li et al., 2005, 2006; Sliter, 1980). However, to the best of our knowledge, there are no publications where analytical methods are applied to the analysis of the effect of layering. The main goal of the present study is to close this gap.

## 2. Mathematical models of the layered shield

### 2.1. Model based on an equation for impact resistance

Consider high speed normal penetration by a rigid sharp striker into a concrete shield. Hereafter, the coordinate  $h$  denotes the instantaneous depth of penetration (DOP) which is defined as the distance between the leading edge of the nose of the impactor and the front surface of the shield. According to an approximate theory of penetration which takes into account only the local impact effects (Kennedy, 1976), the instantaneous resistance force is a function of the depth of penetration  $h$  and the instantaneous impactor velocity  $v$  and can be written in the following form:

$$R(h, v) = g(h)f(v), \quad (1)$$

where  $g(h)$  and  $f(v)$  are functions determining the adopted penetration model. Particular expressions for these functions are suggested by Kennedy (1976), Degen (1980), and Riera (1989).

The equation of motion of the impactor,  $m(d^2h/dt^2) = -R$ , can be rewritten as follows:

$$mv \left( \frac{dv}{dh} \right) = -g(h)f(v), \quad (2)$$

where the instantaneous velocity of the impactor  $v$  is considered to be a function of  $h$ , and  $m$  is the mass of the impactor.

The model determined by Eqs. (1) and (2) was suggested to describe penetration into a semi-infinite shield. In order to adapt

\* Corresponding author.

E-mail address: [elperin@bgu.ac.il](mailto:elperin@bgu.ac.il) (T. Elperin).

### Nomenclature

$b$	depth when perforation occurs; thickness of a shield
$\hat{b}$	depth of penetration
$b^{(j)}$	depth when perforation of the $j$ th plate occurs; thickness of the $j$ th plate
BLV	ballistic limit velocity
$D$	diameter of the cylindrical part of the impactor
DOP	depth of penetration
$e$	perforation thickness of a finite shield normalized by $D$
$E$	energy dissipated during perforation
$f$	function in the expression for resistance force in Eq. (1)
$F$	function in Eq. (6) or $F(z) = z^2$
$g$	function in the expression for resistance force in Eq. (1)
$G$	function in Eq. (6) or function in the model given by Eqs. (5) and (7)
$G^*$	function in Eq. (20)
$h$	instantaneous depth of penetration
$H^{(v\mu)}, H^{(v)}$	elements of the Hessian
$k$	parameter of the model in Eq. (3)
$m$	mass of the impactor
$N$	number of plates in the layered shield
NDRC	National Defense Research Committee
$R$	resistance force in Eq. (1)
$t$	time
$T$	thickness of a shield
$\hat{T}$	perforation thickness
$T_{sum}$	total thickness of the layers in the shield
$T^{(j)}$	thickness of the $j$ th plate
$v$	instantaneous velocity of the impactor
$v_{bl}$	ballistic limit velocity of the monolithic shield
$v_{imp}$	impact velocity of the monolithic shield
$v_{res}$	residual velocity of the monolithic shield
$V_{bl}$	ballistic limit velocity of the layered shield
$V_{imp}$	impact velocity of the layered shield
$V_{res}$	residual velocity of the layered shield
$V_{imp}^{(j)}$	impact velocity of the $j$ th plate
$V_{res}^{(j)}$	residual velocity of the $j$ th plate
$x$	depth of penetration into semi-infinite shield
$z^*$	parameter in Eq. (20)

### Greek letters

$\Delta$	difference, $\Delta = F(V_{bl}) - F(v_{bl})$ in Eq. (29)
$\tilde{\Lambda}$	criterion of optimization in Eq. (33)
$\Lambda$	criterion of optimization in Eq. (35)
$\mu$	parameter in Eq. (10)
$\Psi$	function determining the model in Eq. (10)
$\omega$	function determining the model in Eq. (10)

this model for a shield having a finite thickness, we assume that Eq. (2) describes motion of the impactor until arriving at the depth  $h = b$  where perforation occurs. Note that because of scabbing,  $b$  is smaller than the thickness of the shield,  $T$ . We assume that

$$b = kT, \quad (3)$$

where  $k < 1$ . Formula (3) is similar to the relation between the perforation thickness  $\hat{T}$ , the maximum thickness of the shield which is completely penetrated by an impactor with a given impact velocity, and the reference depth of penetration  $\hat{b}$ , the depth of penetration into a semi-infinite shield by penetrator with the same impact

velocity. In particular, according to the modified Petry's formula this relation is  $\hat{b}/\hat{T} = 0.5$  (Kennedy, 1976).

We consider a range of impact velocities  $v_{imp}$  whereby the projectile perforates the shield. Integrating Eq. (2) by separation of the variables from the initial conditions at the impact ( $h = 0$ ,  $v = v_{imp}$ ) to the final conditions corresponding to penetrating at the depth  $h = b = kT$  with the velocity  $v_{res}$ , we obtain the following relationship:

$$m \int_{v_{imp}}^{v_{res}} \frac{v dv}{f(v)} = - \int_0^b g(h) dh. \quad (4)$$

Eq. (4) can be rewritten as follows:

$$F(v_{imp}) - F(v_{res}) = G(b), \quad (5)$$

where

$$F(z) = m \int_0^z \frac{v dv}{f(v)}, \quad G(z) = \int_0^z g(h) dh. \quad (6)$$

The ballistic limit velocity (BLV)  $v_{bl}$  is defined as the minimum impact velocity of the impactor required for perforating the shield. Assuming that perforation occurs when the impactor emerges from the shield with a zero residual velocity and taking into account that  $F(0) = 0$ , Eq. (5) implies that

$$F(v_{bl}) = G(b). \quad (7)$$

The shortcomings of the model determined by Eqs. (1) and (3) are associated with physical interpretation of function  $g(h)$  and the lack of validation of this model for finite thickness shields.

On the other hand, describing the motion of a projectile inside the shield is redundant in this study since what is needed is only the relationship between the impact velocity and the residual velocity of the impactor. In the following we validate the model determined by Eqs. (5) and (7) with  $F(z) = z^2$  and  $k = 1$  which without using the resistance model.

### 2.2. Model based on the equation of energy conservation

Based on the frequently used assumption (see, e.g., Barr, 1990; Li et al., 2006) that energy dissipated during perforation,  $E$ , is independent of the impact velocity, equation of energy conservation can be written as  $0.5mv_{imp}^2 - 0.5mv_{res}^2 = E$ . Since  $v_{imp} = v_{bl}$  if  $v_{res} = 0$ , it can be concluded that  $E = 0.5mv_{bl}^2$  and, consequently, the following relationship between the impact, residual and ballistic limit velocities is valid:

$$v_{imp}^2 - v_{res}^2 = v_{bl}^2. \quad (8)$$

Eq. (8) coincides with Eq. (5) when

$$F(z) = z^2, \quad k = 1 (b = T). \quad (9)$$

For studying the behavior of the function  $G(T) = G(b)$ , we use well known semi-empirical models describing the dependence between  $v_{bl}^2$  and the thickness of the shield (Adeli and Amin, 1985; Li et al., 2005). After some algebra, corresponding formulas can be represented in the following form that is convenient for our analysis:

$$v_{bl}^2 = \mu \Psi(x), \quad x = \omega(e), \quad (10)$$

where  $\mu$  is parameter which generally depends on the mechanical properties of the material of the shield, mass and diameter of the impactor,  $D$ ,  $\Psi$  and  $\omega$  are functions determining the model,  $x$  and  $e$  are the depth of penetration into a semi-infinite shield and the perforation thickness of a finite shield normalized by  $D$ , correspondingly.

The particular expressions for the functions  $\Psi$  and  $\omega$  for a number of suggested models are summarized as follows:

(1) Ballistic Research Laboratory (BRL) model:

$$\Psi(x) = x^{1.5}, \quad \omega(e) = 0.769e; \quad (11)$$

(2) Army Corps of Engineers (ACE) model:

$$\Psi(x) = (x - 0.5)^{1.33}, \quad \omega(e) = 0.81e - 1.06, \quad 3 \leq e \leq 18; \quad (12)$$

(3) Modified National Defense Research Committee (NDRC) model:

$$\Psi(x) = \begin{cases} (x/2)^{2.22} & \text{if } x \leq 2 \\ (x-1)^{1.11} & \text{if } x > 2 \end{cases}, \quad (13)$$

$$\omega(e) = \begin{cases} 2.22 - \sqrt{4.94 - 1.39e} & \text{if } e \leq 3 \\ 0.81e - 1.06 & \text{if } 3 \leq e \leq 18 \end{cases}; \quad (14)$$

(4) Degen's model:

$$\Psi(x) = \begin{cases} (x/2)^{2.22} & \text{if } x \leq 2 \\ (x-1)^{1.11} & \text{if } x > 2 \end{cases}, \quad (15)$$

$$\omega(e) = \begin{cases} 3.67 - \sqrt{13.44 - 3.33e} & \text{if } e \leq 2.65 \\ 0.77e - 0.53 & \text{if } 2.65 \leq e \leq 18 \end{cases}; \quad (16)$$

(5) Chang's and Crieri's models:

$$v_{bl}^2 = \mu e^2; \quad (17)$$

(6) Haldar–Miller model:

$$\Psi(x) = \begin{cases} 4.541x + 0.124 & \text{if } 0.0388 \leq x < 0.5234 \\ 2.242x + 1.327 & \text{if } 0.5234 < x \leq 0.746 \\ 14.51x - 7.819 & \text{if } 0.746 < x < 1.986 \end{cases},$$

$$\omega(e) = \begin{cases} 2.22 - \sqrt{4.94 - 1.39e} & \text{if } e \leq 3 \\ 0.81e - 1.06 & \text{if } 3 \leq e \leq 18 \end{cases}; \quad (18)$$

(7) Hughes' model:

$$\Psi(x) = x, \quad \omega(e) = \begin{cases} 0.28e & \text{if } e \leq 2.52 \\ 0.63e - 0.89 & \text{if } e > 2.52 \end{cases}. \quad (19)$$

Fig. 1a and b shows normalized squared ballistic limit velocity,  $\bar{v}_{bl}^2 = v_{bl}^2(e)/v_{bl}^2(5) = \Psi(\omega(e))/\Psi(\omega(5))$ , as a function of the dimensionless thickness of the shield,  $e$ , calculated using Eqs. (11)–(19). It must be noted that these models are devised without requiring the smoothness of the approximating functions in the endpoints of the neighboring intervals. Consequently, it can be concluded that, in the general case, the dependence  $\bar{v}_{bl}^2 = \bar{v}_{bl}^2(e)$ , is described by an increasing concave function which approaches a linear function for large  $e$  (e.g., in the case of the Haldar–Miller model).

Hereafter, we use the model given by Eqs. (5) and (7) and assume that the function  $G(z)$  has, generally, the following form:

$$G(z) = \begin{cases} G_*(z) & \text{if } 0 \leq z \leq z_* \\ \alpha_1 z + \alpha_0 & \text{if } z \geq z_* \end{cases}, \quad G(0) = 0, \quad (20)$$

where  $G_*(z)$  is an increasing concave function for  $0 \leq z \leq z_*$ ,  $G''_*(z_*) = 0$ , coefficients  $\alpha_0$  and  $\alpha_1$  satisfy the conditions  $\alpha_1 z_* + \alpha_0 = G_*(z_*)$  and  $\alpha_1 = G'_*(z_*)$ .

### 2.3. Model of a layered shield

We study the ballistic properties of the multilayered shields under the assumption that the layers are perforated sequentially and independently, i.e., either the interaction between the layers is neglected or it is assumed that there are relatively wide air gaps between the layers. Hence the residual velocity of the  $j$ th plate,  $V_{res}^{(j)}$ , is equal to the impact velocity of the  $(j+1)$ th plate,  $V_{imp}^{(j+1)}$ , for  $j=1 \div N-1$ . In addition, the impact velocity of the shield  $V_{imp}$  is

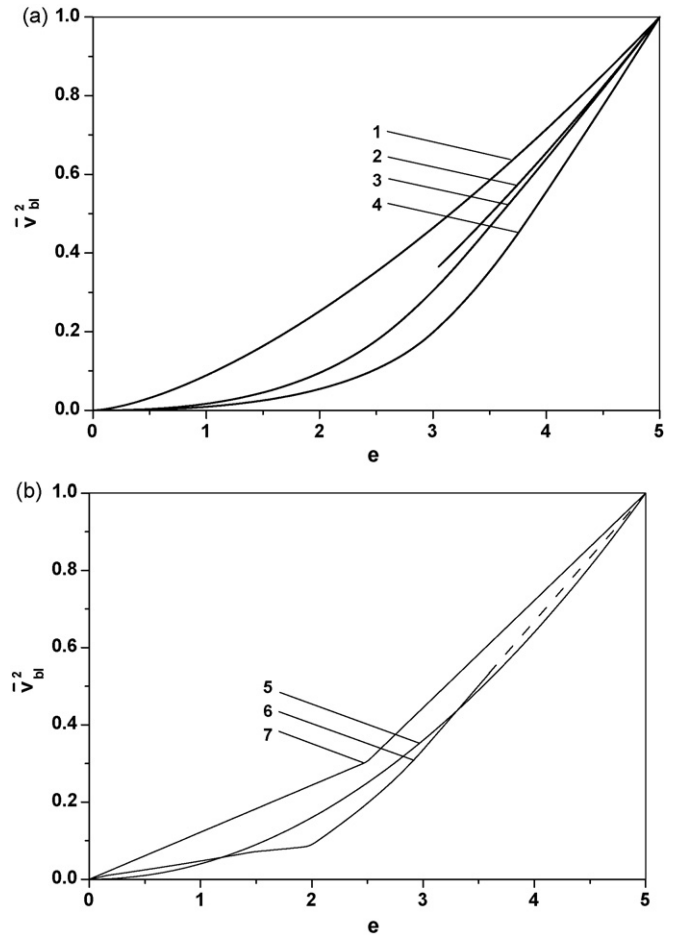


Fig. 1. (a and b) Normalized squared BLV as a function of normalized thickness of the shield calculated using different models: 1: BRL; 2: ACE; 3: NDRC; 4: Degen; 5: Chang and Crieri; 6: Haldar–Miller; 7: Hughes.

equal to  $V_{imp}^{(1)}$ , and the residual velocity of the shield  $V_{res}$  is equal to  $V_{res}^{(N)} = V_{imp}^{(N+1)}$  (the notation  $V_{imp}^{(N+1)}$  is introduced here for convenience). Consequently, the mathematical model of the layered plate can be written in the following form:

$$F(V_{imp}^{(j)}) - F(V_{imp}^{(j+1)}) = G(kT^{(j)}), \quad j = 1, 2, \dots, N, \quad (21)$$

where

$$V_{imp}^{(1)} = V_{imp}, \quad V_{imp}^{(N+1)} = V_{res}. \quad (22)$$

Hereafter, superscript in the brackets denotes the ordinal number of the layer.

### 3. Comparison of the monolithic and layered shields

In this section, we compare the magnitudes of the BLVs of the layered shield and of the monolithic shield having the same thickness.

Eq. (21) written for  $j=1, 2, \dots, N$  and Eq. (22) yield:

$$F(V_{imp}) - F(V_{res}) = \sum_{j=1}^N G(kT^{(j)}). \quad (23)$$

Equation for the BLV of the layered shield,  $V_{bl}$ , is obtained from Eq. (23) after setting  $V_{imp} = V_{bl}$  and  $V_{res} = 0$ :

$$F(V_{bl}) = \sum_{j=1}^N G(kT^{(j)}). \quad (24)$$

The latter equation implies that in the framework of the considered model, the rearranging of the plates in a multilayered shield does not affect the BLV.

In the case of the monolithic shield with the thickness

$$T_{sum} = T^{(1)} + T^{(1)} + \dots + T^{(N)}, \quad (25)$$

the relationship between its BLV,  $v_{bl}$ , and the thickness of the shield can be obtained from Eq. (24) setting  $V_{bl} = v_{bl}$ ,  $N = 1$  and  $T^{(j)} = T_{sum}$ :

$$F(v_{bl}) = G(kT_{sum}). \quad (26)$$

Taking into account Eq. (25) we can rewrite Eqs. (24) and (26) as follows:

$$F(V_{bl}) = \sum_{j=1}^N G(b^{(j)}), \quad F(v_{bl}) = G\left(\sum_{j=1}^N b^{(j)}\right), \quad (27)$$

$$b^{(j)} = kT^{(j)}, \quad j = 1, 2, \dots, N.$$

Consider the following question: when layering does not affect the BLV of the shield? This occurs if  $F(V_{bl}) = F(v_{bl})$  for every set  $b^{(1)}, b^{(2)}, \dots, b^{(N)}$ , i.e., the following equation:

$$\sum_{j=1}^N G(b^{(j)}) = G\left(\sum_{j=1}^N b^{(j)}\right). \quad (28)$$

must be valid for arbitrary set of  $b^{(1)}, b^{(2)}, \dots, b^{(N)}$ . The solution of the functional equation (28) is  $G(z) = cz$ , where  $c$  is a constant (see Aczél and Dhombres, 1989). Eq. (6) implies that  $g(z) = G'(z) = c$  in this case.

Therefore, the BLV of the shield does not change under any layering if and only if the resistance in Eq. (1) does not depend on the instantaneous DOP (the model in Section 2.1) or the BLV of a shield is proportional to the squared thickness of the shield (the model in Section 2.2).

Now let us compare the monolithic and layered shields having the same thickness. Since  $F(z)$  is an increasing function, we can consider the difference of the functions for two values of the BLV:

$$\Delta = F(V_{bl}) - F(v_{bl}) = \sum_{j=1}^N G(b^{(j)}) - G\left(\sum_{j=1}^N b^{(j)}\right), \quad (29)$$

rather than the difference of the BLVs,  $V_{bl} - v_{bl}$ .

Introduce the following notation:  $s_i = b^{(1)} + b^{(2)} + \dots + b^{(i)}$ . After some algebra we obtain:

$$\begin{aligned} G\left(\sum_{j=1}^N b^{(j)}\right) &= \int_0^{s_N} G'(z) dz \\ &= \int_0^{b^{(1)}} G'(z) dz + \sum_{j=2}^N \int_{s_{j-1}}^{s_{j-1}+b^{(j)}} G'(z) dz \\ &= G(b^{(1)}) + \sum_{j=2}^N \int_0^{b^{(j)}} G'(z + s_{j-1}) dz, \end{aligned} \quad (30)$$

$$\sum_{j=1}^N G(b^{(j)}) = \sum_{j=2}^N \int_0^{b^{(j)}} G'(z) dz. \quad (31)$$

Eqs. (30) and (31) imply that

$$\Delta = \sum_{j=2}^N \int_0^{b^{(j)}} [G'(z) - G'(z + s_{j-1})] dz. \quad (32)$$

Examining Eq. (32) allows us to arrive at the following conclusion. If a non decreasing in the interval  $[0, b^{(1)} + b^{(2)} + \dots + b^{(N)}]$  function  $G'(z)$  increases in some sub-interval of this interval, then  $\Delta < 0$ , and the monolithic shield offers an advantage over any layered shield. In other words, this occurs if a non-convex function  $G(z)$  is concave in some sub-interval.

## 4. Effect of layering on ballistic properties of shields

### 4.1. Formulation of the problem

According to the above analysis, ballistic properties of the shield may be impaired by layering. Let us estimate the maximum reduction in the ballistic performance of the shield. To do this end, consider the following problem: how one can split a monolithic shield into plates in order to obtain the minimum BLV of the layered shield. We assume here that the number of the plates in the shield is given. The first equation in Eq. (27) implies that the problem reduces to the minimization of the following function:

$$\tilde{\Lambda}(b^{(1)}, b^{(2)}, \dots, b^{(N)}) = \sum_{j=1}^N G(b^{(j)}). \quad (33)$$

Minimization is performed under the following constraints:

$$\sum_{j=1}^N b^{(j)} = b_{sum}, \quad b^{(i)} > 0, \quad i = 1, \dots, N, \quad (34)$$

where the total thickness of the plates  $b_{sum}$  is given. In the following we consider an equivalent formulation of this problem ( $b_{sum} = kT_{sum}$ ): minimization of the function

$$\Lambda(b^{(1)}, \dots, b^{(N-1)}) = \sum_{j=1}^{N-1} G(b^{(j)}) + G\left(b_{sum} - \sum_{j=1}^{N-1} b^{(j)}\right) \rightarrow \min. \quad (35)$$

Minimization is performed under the following constraints:

$$\sum_{j=1}^{N-1} b^{(j)} < b_{sum}, \quad b^{(i)} > 0, \quad i = 1, \dots, N-1. \quad (36)$$

### 4.2. The case $N=2$

First let us consider the most simple and the most practically interesting case of two layers. In this case Eqs. (35) and (36) can be written as follows:

$$\Lambda(b^{(1)}) = G(b^{(1)}) + G(b_{sum} - b^{(1)}) \rightarrow \min, \quad 0 < b^{(1)} < b_{sum}. \quad (37)$$

Let us calculate the derivatives:

$$\begin{aligned} \Lambda'(b^{(1)}) &= G'(b^{(1)}) - G'(b_{sum} - b^{(1)}), \\ \Lambda''(b^{(1)}) &= G''(b^{(1)}) + G''(b_{sum} - b^{(1)}). \end{aligned} \quad (38)$$

If the interval with the constant value of  $G'(h) = G'_*(z_*)$  does not exist ( $z_* = \infty$ ), then

$$\begin{aligned} \Lambda'(b^{(1)}) &= G'_*(b^{(1)}) - G'_*(b_{sum} - b^{(1)}), \\ \Lambda''(b^{(1)}) &= G''_*(b^{(1)}) + G''_*(b_{sum} - b^{(1)}). \end{aligned} \quad (39)$$

Since  $G'_*(z)$  is an increasing function for  $z \geq 0$ , the condition for the extremum,  $\Lambda'(b^{(1)}) = 0$ , can be satisfied only when  $b^{(1)} = b_{sum} - b^{(1)}$ , i.e., equation  $\Lambda'(b^{(1)}) = 0$  has a single root,  $b^{(1)} = 0.5b_{sum}$ . The latter value provides minimum of  $\Lambda'(b^{(1)})$  since  $\Lambda''(0.5b_{sum}) = 2G''_*(0.5b_{sum}) > 0$ . Therefore, the minimum BLV is attained when the monolithic plate is separated into two plates having equal thicknesses.

Assume now that  $z_* < \infty$ , i.e., the interval with the constant value of  $G'(z) = G'_*(z_*)$  does exist. Let us show that the equation  $g(b^{(1)}) = g(b^{(2)})$  can be satisfied only in two cases:  $b^{(1)} < z_*$  and  $b^{(2)} < z_*$  or  $b^{(1)} > z_*$  and  $b^{(2)} > z_*$ . The proof is given by the following two sequences of inequalities: the first one,  $b^{(1)} < z_* \Rightarrow G'(b^{(1)}) < G'(z_*) \Rightarrow G'(b^{(2)}) < G'(z_*) \Rightarrow b^{(2)} < z_*$  and the second one,  $b^{(1)} > z_* \Rightarrow G'(b^{(1)}) = G'(z_*) \Rightarrow G'(b^{(2)}) = G'(z_*) \Rightarrow b^{(2)} > z_*$ .

If  $b_{sum} < 2z_*$ , then only the first possibility ( $b^{(1)} < z_*$  and  $b^{(2)} < z_*$ ) can be realized in the point of the extremum of  $\Lambda(b^{(1)})$ . Consequently, Eq. (39) has the form of Eq. (38) and the single minimum of  $\Lambda(b^{(1)})$  is attained for  $b^{(1)} = 0.5b_{sum}$ .

If  $b_{sum} > 2z_*$ , then the condition of extremum,  $\Lambda'(b^{(1)}) = 0$ , implies the inequalities  $b^{(1)} > z_*$  and  $b^{(2)} = b_{sum} - b^{(1)} > z_*$ . Since  $\Lambda'' = 0$  for such  $b^{(1)}$ , it is convenient to consider the following relationship for  $G'(b^{(1)})$ :

$$\begin{aligned} G'(b^{(1)}) &= \Lambda'(b^{(1)}) \\ &= \begin{cases} G'_*(b^{(1)}) - G'_*(z_*) < 0 & \text{if } 0 < b^{(1)} < z_* \\ 0 & \text{if } z_* \leq b^{(1)} \leq b_{sum} - z_* \\ G'_*(z_*) - G'_*(b_{sum} - b^{(1)}) > 0 & \text{if } b_{sum} - z_* < b^{(1)} < b_{sum} \end{cases}. \end{aligned} \quad (40)$$

Function  $\Lambda'(b^{(1)})$  decreases when  $0 < b^{(1)} < z_*$ , remains constant when  $z_* \leq b^{(1)} \leq b_{sum} - z_*$  and increases when  $b_{sum} - z_* < b^{(1)} < b_{sum}$ . Since  $z_* \leq 0.5b_{sum} \leq b_{sum} - z_*$ , maximum of  $\Lambda'(b^{(1)})$  equals to  $\Lambda(0.5b_{sum})$ .

Consequently, we established and proved the following assertion. If  $b_{sum} \leq 2z_*$ , then the minimum BLV is attained only when the monolithic plate is separated into two plates having equal thicknesses. If  $b_{sum} > 2z_*$  then the minimum BLV is attained for all  $z_* < b^{(1)} < b_{sum}$  and  $z_* < b^{(2)} \leq b_{sum}$  including the particular case of splitting the shield into two plates with equal thicknesses.

#### 4.3. General case

Let us investigate the case with arbitrary number of layers. Since the first and the second order derivatives of  $G$  exist,  $\Lambda$  is a sufficiently smooth function. The equations for the first and the second order derivatives of  $\Lambda$  read:

$$H^{(v)} \equiv \frac{\partial \Lambda}{\partial b^{(v)}} = G'(b^{(v)}) - G'(b^{(N)}), \quad (41)$$

$$H^{(v,\mu)} \equiv \frac{\partial^2 \Lambda}{\partial b^{(v)} \partial b^{(\mu)}} = \delta_{v\mu} G''(b^{(v)}) + G''(b^{(N)}), \quad (42)$$

where  $v = 1, \dots, N-1$ ,  $\mu = 1, \dots, N-1$ ,

$$\delta_{v\mu} = \begin{cases} 1 & \text{if } v = \mu \\ 0 & \text{if } v \neq \mu \end{cases}, \quad b^{(N)} = b_{sum} - \sum_{j=1}^{N-1} b^{(j)}. \quad (43)$$

The Hessian (the matrix with the elements  $H^{(v\mu)}$ ) plays the role of the second derivative of  $\Lambda$ , considered as a function of many variables (see, e.g., Korn and Korn, 1968). Properties of the Hessian are

associated with the convexity of the function  $\Lambda$ , and they depend on the signs of the roots of the characteristic equation:

$$\begin{vmatrix} H^{(1,1)} - \eta & H^{(1,2)} & \dots & H^{(1,N-1)} \\ H^{(2,1)} & H^{(2,2)} - \eta & \dots & H^{(2,N-1)} \\ \dots & \dots & \dots & \dots \\ H^{(N-1,1)} & H^{(N-1,2)} & \dots & H^{(N-1,N-1)} - \eta \end{vmatrix} = 0. \quad (44)$$

Since the last term in Eq. (42) is the same for all  $H^{(v,\mu)}$ , this term can be dropped out in the determinant in Eq. (44). Nonzero elements remain only at the diagonal, and the characteristic equation reads:

$$\prod_{j=1}^{N-1} [G''(b^{(j)}) - \eta] = 0. \quad (45)$$

The domain determined by the linear inequalities (36) is convex. Since  $G''(z) \geq 0$  in this domain, all the roots of Eq. (45) are nonnegative, and hence the Hessian is positive-semidefinite. In classical mathematical analysis such functions are known as non-convex functions.

Let us prove that the considered problem always has a unique solution if

$$b_{sum} < Nz_*. \quad (46)$$

Using Eq. (41), let us rewrite the conditions  $H^{(v)} = 0$  ( $v = 1, \dots, N-1$ ) in the following form:

$$G'(b^{(1)}) = G'(b^{(2)}) = \dots = G'(b^{(N-1)}) = G'(b^{(N)}). \quad (47)$$

Eq. (46) implies that there exists at least one  $b^{(v_*)} < z_*$ . Taking into account the property of the function  $G(z)$  we conclude, similar to the case  $N=2$ , that all  $b^{(v)} \leq z_*$  and Eq. (47) have unique solution:

$$\tilde{b}^{(1)} = \tilde{b}^{(2)} = \dots = \tilde{b}^{(N-1)} = \tilde{b}^{(N)} = \frac{b_{max}}{N}. \quad (48)$$

Since all  $\tilde{b}^{(v)} < z_*$ , then  $G''(\tilde{b}^{(v)}) > 0$ , Eq. (45) has only positive roots and the Hessian is positive-definite. Therefore,  $\Lambda$  is strictly concave at the point  $b^{(v)} = \tilde{b}^{(v)}$ , and this point is the point of a local minimum. In convex programming (the considered problem belongs to the class of problems considered in convex programming), a local minimum (if it does exist) is also a global minimum (see, e.g., Rockafellar, 1970). Therefore, it can be concluded that the global minimum of function  $\Lambda$  is attained at the point determined by Eq. (48), i.e., when the thicknesses of the plates are equal.

In the case when

$$b_{sum} > Nz_*, \quad (49)$$

there exists a non-empty sub-domain of the domain determined by Eq. (36) that is associated with values  $b^{(v)}$  for which the BLV is minimum. This sub-domain is determined by the following inequities:

$$z_* < b^{(v)} \leq b_{sum}, \quad \sum_{j=1}^{N-1} b^{(j)} < b_{sum} - z_*, \quad v = 1, \dots, N-1. \quad (50)$$

The structure of this sub-domain for  $N=3$  is showed in Fig. 2.

Fig. 3 shows maximal decrease of the BLV as a function of the number of layers for the NDRC model. In order to quantify the decrease of the BLV we use the normalized BLV,  $v_{bl}(N)/v_{bl}(1)$ , where  $v_{bl}(N)$  is the BLV of the layered shield with  $N$  equal layers and  $v_{bl}(1)$  is the BLV of the monolithic shield with the same thickness. When  $v_{bl}^2$  is proportional to  $e^n$  ( $n=1.5$  for the BRL model,  $n=2$  the Chang's model and Crieri's model), the following simple relationship can be obtained:

$$\frac{v_{bl}(N)}{v_{bl}(1)} = N^{(1-n)/2}. \quad (51)$$



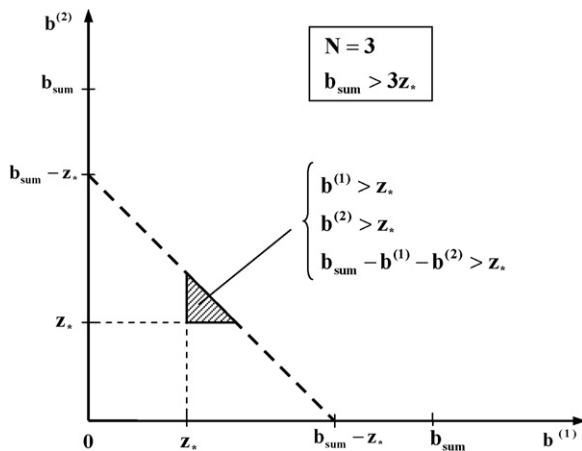


Fig. 2. Sub-domain of the thicknesses of the plates for a three-plate shield with values of  $b^{(1)}$  and  $b^{(2)}$  that provide the largest reduction of BLV by layering.

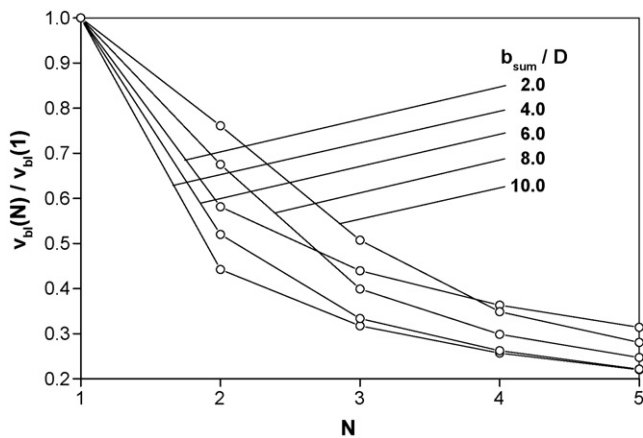


Fig. 3. Decrease of the BLV with increasing the number of layers for the NDRC model.

Eq. (51) and Fig. 3 show that layering can cause a considerable reduction of the ballistic performance of a concrete shield. Clearly, the quantitative results obtained using the approximate models can be of interest primarily as a basis for qualitative conclusions.

## 5. Concluding remarks

Using a wide class of semi-empirical models, we investigated the effect of layering of concrete shields and established a number of general assertions concerning ballistic properties of the shields. Omitting the technical details, the main conclusions of this study can be formulated as follows:

- (1) The BLV of the multilayered shield does not depend on the order of the plates in the shield.
- (2) Monolithic shield performs better than any layered shield with the same thickness.
- (3) The largest decrease of the BLV occurs when the shield is divided into the plates having the same thickness.

Note that we did not specify a particular relationship between the BLV and the thickness of the shield, but used only some general properties of this dependence.

Clearly, the obtained theoretical predictions require experimental validation.

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